Projective isomorphisms between rational surfaces

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The Problem

- If f is a rational map, then we denote its domain by dom f and the Zariski closure of its image by img f.
- **Problem.** Given birational maps $f\colon \operatorname{dom} f \dashrightarrow \mathbb{P}^n$ and $g\colon \operatorname{dom} g \dashrightarrow \mathbb{P}^n$ such that $\operatorname{dom} f, \operatorname{dom} g \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}$ and $n > 2$, determine the set of projective isomorphisms between their images:

$$\mathcal{P}(f,g) := \{\rho \in \operatorname{Aut}\mathbb{P}^n \mid \rho(\operatorname{img} f) = \operatorname{img} g\}.$$

- **Example.**

$$f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^3, (x_0 : x_1 : x_2) \mapsto (x_0^6 x_1^2 : x_0 x_1^5 x_2^2 : x_1^3 x_2^5 : x_0^5 x_1 x_2^2 + 2 x_0^5 x_2^3),$$

$$g\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3, \quad (y_0 : y_1; y_2 : y_3) \mapsto (y_0^3 y_1^2 y_2^5 : y_0^3 y_1^2 y_2^5 + y_1^5 y_2^3 y_3^2 : y_0^2 y_1^3 y_3^5 : y_0^4 y_1 y_2^3 y_3^2 + y_0^5 y_2^2 y_3^3 + y_0^2 y_1^3 y_3^5).$$

$\mathcal{P}(f,g) \subset \operatorname{Aut}\mathbb{P}^3$ is up to a scalar factor represented by the following parametrized matrix with $\alpha \neq 0$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4\alpha^5 & 0 & 0 \\ 0 & 0 & 32\alpha^6 & 0 \\ 0 & 0 & 32\alpha^6 & 4\alpha \end{bmatrix}$$

Preliminaries

- **Projective space** $\mathbb{P}^n$ is defined as $\mathbb{C}^n \setminus \{(0,\ldots,0)\}/\sim$, where $(x_0,\ldots,x_n) \sim (\lambda\, x_0,\ldots,\lambda\, x_n)$ for all non-zero $\lambda \in \mathbb{C}^*$.
- A rational map $f\colon \operatorname{dom} f \dashrightarrow \mathbb{P}^n$ is called **birational** if it is almost everywhere an isomorphism.
- **Example.** The following rational map is not defined if $x\,y = 0$:

  $$t\colon \mathbb{C}^2 \dashrightarrow \mathbb{C}^2, \quad (x,y) \mapsto (\tfrac{1}{x}, \tfrac{1}{y})$$

  The dashed arrow indicates that the map is not everywhere defined.
  Via the embedding $\iota\colon \mathbb{C}^2 \hookrightarrow \mathbb{P}^2,\ (x,y) \mapsto (1 : x : y)$ we can represent the map t projectively in terms of a tuple of polynomials:

  $$f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1)$$

  Indeed, if $x_0\,x_1\,x_2 \neq 0$ and $x\,y \neq 0$, then

  $$f(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}) = (1 : \frac{x_0}{x_1} : \frac{x_0}{x_2}) \quad \text{and} \quad \iota \circ t(x,y) = (1 : \frac{1}{x} : \frac{1}{y})$$

  The map f is birational, since it is almost everywhere an isomorphism, namely outside the coordinate lines $x_0 = 0$, $x_1 = 0$ and $x_2 = 0$.
- We call $f\colon \operatorname{dom} f \to \mathbb{P}^n$ **regular** if it is everywhere defined.

Compatible Reparametrizations

- A projective isomorphism $\rho\colon \mathbb{P}^n \to \mathbb{P}^n$ such that $\rho(\operatorname{img} f) = \operatorname{img} g$ induces a birational map r between the domains of f and g:

$$\begin{array}{ccc} \operatorname{img} f & \xrightarrow{\rho} & \operatorname{img} g \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ \operatorname{dom} f & \overset{r}{\dashrightarrow} & \operatorname{dom} g \end{array}$$

- $r = g^{-1} \circ \rho \circ f$ is an element of the **compatible reparametrizations**:

$$\mathcal{R}(f,g) := \{ r \in \operatorname{bir}(\operatorname{dom} f, \operatorname{dom} g) \mid \rho \circ f = g \circ r \text{ for some } \rho \in \mathcal{P}(f,g) \},$$

  where $\operatorname{bir}(\operatorname{dom} f, \operatorname{dom} g)$ is the set of birational maps between the domains.

  Recall. $\mathcal{P}(f,g) := \{ \rho \in \operatorname{Aut} \mathbb{P}^n \mid \rho(\operatorname{img} f) = \operatorname{img} g \}$ denotes the set of projective isomorphism between the image surfaces.

- **Strategy.**
  Step 1: Determine a super set $\mathcal{S}$ of $\mathcal{R}(f,g)$.
  Step 2: Recover $\mathcal{P}(f,g)$ from $\mathcal{S}$.

Recover $\mathcal{P}(f,g)$ from $\mathcal{S} \supset \mathcal{R}(f,g)$: part 1/3

- **Example.** Suppose that $g := f$, where

$$f\colon \mathbb{P}^2 \to \mathbb{P}^3, x \mapsto (x_0^2 + x_1^2 + x_2^2 : x_0\,x_1 : x_0\,x_2 : x_1\,x_2)$$

Since f and g are both regular (thus everywhere defined) it follows that

$$\mathcal{S} := \{r_c\}_{c\in\mathcal{I}} \cong \operatorname{Aut}\mathbb{P}^2 \quad \text{is a super set of} \quad \mathcal{R}(f,g),$$

where an isomorphism $r_c\colon \mathbb{P}^2 \to \mathbb{P}^2$ is a linear map of the form

$$x \mapsto (c_0\,x_0 + c_1\,x_1 + c_2\,x_2 : c_3\,x_0 + c_4\,x_1 + c_5\,x_2 : c_6\,x_0 + c_7\,x_1 + c_8\,x_2),$$

and the index set is represented as

$$\mathcal{I} := \Big\{ c \in \mathbb{F}^9 \;:\; \text{the matrix } \left(\begin{smallmatrix} c_0 & c_1 & c_2 \\ c_3 & c_4 & c_5 \\ c_6 & c_7 & c_8 \end{smallmatrix}\right) \text{ has non-zero determinant and its first non-zero entry is equal to one} \Big\}.$$

- **Question.** How can we recover projective isomorphisms $\mathcal{P}(f,g)$ from a super set $\mathcal{S}$ of the compatible reparametrizations?

Recover $\mathcal{P}(f,g)$ from $\mathcal{S} \supset \mathcal{R}(f,g)$: part 2/3

- Let $\mathcal{M}$ be defined as the set of all rational maps $f\colon \operatorname{dom} f \dashrightarrow \operatorname{img} f \subseteq \mathbb{P}^{\dim f}$ such that $\operatorname{dom} f \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}$.

- Let M_f denote the **coefficient matrix** of $f \in \mathcal{M}$ with respect to the standard basis for monomials of (bi)-degree equal to the components of f.

- **Example.** The standard monomial basis for quadratic maps is

$$(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2),$$

If $f\colon \mathbb{P}^2 \to \mathbb{P}^3, x \mapsto (x_0^2 + x_1^2 + x_2^2 : x_0\, x_1 : x_0\, x_2 : x_1\, x_2)$, then

$$M_f = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

- In the next slide we explain with an example how to recover the projective isomorphisms $\mathcal{P}(f,g)$ from a super set $\mathcal{S}$ of the compatible reparametrizations $\mathcal{R}(f,g)$.

- **Remark.** The idea in the next slide works in the general setting, although then we also have to take "base points" into account, which we will consider at slide 9.

Recover $\mathcal{P}(f,g)$ from $\mathcal{S} \supset \mathcal{R}(f,g)$: part 3/3

We denote an identity matrix and zero matrix by $\mathbf{1}$ and $\mathbf{0}$, respectively.

Let $\ker M_f$ be a matrix whose columns form a basis for the kernel of M_f.

Let $\chi_U : \mathbb{P}^n \to \mathbb{P}^n$ denote the projective linear map corresponding to matrix U.

As before suppose that $g := f$ and

$$f : \mathbb{P}^2 \to \mathbb{P}^3, \quad x \mapsto (x_0^2 + x_1^2 + x_2^2 : x_0 x_1 : x_0 x_2 : x_1 x_2),$$

Recall that $\mathcal{S} := \{r_c\}_{c \in \mathcal{I}} \cong \operatorname{Aut} \mathbb{P}^2$ is in this case a super set of $\mathcal{R}(f,g)$.

Proposition. $\mathcal{P}(f,g) = \{\chi_U \in \operatorname{Aut} \mathbb{P}^n \mid U \oplus \mathbf{1} = E_{g \circ r_c} \cdot E_f^{-1} \text{ and } c \in \mathcal{J}\}$,

where

$$E_f = \left(M_f^\top \mid \ker M_f\right)^\top, \qquad E_{g \circ r_c} = \left(M_{g \circ r_c}^\top \mid \ker M_f\right)^\top,$$

$$M_f = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \end{pmatrix}, \qquad (\ker M_f)^\top = \begin{pmatrix} 1&0&0&0&0&-1 \\ 0&0&0&1&0&-1 \end{pmatrix}, \qquad M_{g \circ r_c} =$$

$$\begin{pmatrix} c_0^2+c_3^2+c_6^2 & 2c_0c_1+2c_3c_4+2c_6c_7 & 2c_0c_2+2c_3c_5+2c_6c_8 & c_1^2+c_4^2+c_7^2 & 2c_1c_2+2c_4c_5+2c_7c_8 & c_2^2+c_5^2+c_8^2 \\ c_0c_3 & c_1c_3+c_0c_4 & c_2c_3+c_0c_5 & c_1c_4 & c_2c_4+c_1c_5 & c_2c_5 \\ c_0c_6 & c_1c_6+c_0c_7 & c_2c_6+c_0c_8 & c_1c_7 & c_2c_7+c_1c_8 & c_2c_8 \\ c_3c_6 & c_4c_6+c_3c_7 & c_5c_6+c_3c_8 & c_4c_7 & c_5c_7+c_4c_8 & c_5c_8 \end{pmatrix}$$

and

$$\mathcal{J} = \{c \in \mathcal{I} \mid M_{g \circ r_c} \cdot \ker M_f = \mathbf{0}\} :$$ "a set of algebraic equations"

Recall

- If f is a rational map into projective space, then
  $\operatorname{dom} f$ denotes its domain, $\operatorname{img} f \subset \mathbb{P}^{\dim f}$ the Zariski closure of its image,
  $\dim f$ denotes the projective embedding dimension.
- **Recall.** Rational maps with domain in $\{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}$:

  $$\mathcal{M} := \{f\colon \operatorname{dom} f \dashrightarrow \operatorname{img} f \subseteq \mathbb{P}^{\dim f} \mid \operatorname{dom} f \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}\}.$$

  The **projective isomorphisms** for $f, g \in \mathcal{M}$:

  $$\mathcal{P}(f,g) := \{\rho \in \operatorname{Aut} \mathbb{P}^{\dim f} \mid \rho(\operatorname{img} f) = \operatorname{img} g\}.$$

  The **compatible reparametrizations** for $f, g \in \mathcal{M}$:

  $$\mathcal{R}(f,g) := \{r \in \operatorname{bir}(\operatorname{dom} f, \operatorname{dom} g) \mid \rho \circ f = g \circ r \text{ for some } \rho \in \mathcal{P}(f,g)\},$$

  where $\operatorname{bir}(\operatorname{dom} f, \operatorname{dom} g)$ is the set of birational maps between the domains.
- **Problem.** For given birational maps $f, g \in \mathcal{M}$ determine $\mathcal{P}(f,g)$.
- **Strategy.**
  Step 1: Determine a super set $\mathcal{S}$ of $\mathcal{R}(f,g)$.
  Step 2: Recover $\mathcal{P}(f,g)$ from $\mathcal{S}$.
- We still need to explain step 1 in case f and g are not regular.

Base points and classes of rational functions 1/2

- Let the rational map $f\colon \operatorname{dom} f \dashrightarrow \mathbb{P}^n$ send x to $(f_0(x) : \ldots : f_n(x))$.

- We call $p \in \operatorname{dom} f$ an m-**fold simple base point** of f if almost all forms in the vector space $\langle f_0, \ldots, f_n \rangle$ vanish with multiplicity $m \in \mathbb{Z}_{>0}$ at p.

  Remark. In general there are also **infinitely near base points**.

- If $\operatorname{dom} f = \mathbb{P}^2$, f has an m_i-fold base point at p_i for $1 \leq i \leq r$ and $d = \deg f_0 = \ldots = \deg f_n$, then the **class** of f is defined as

$$[f] = d\, e_0 - m_1\, e_1 - \ldots - m_r\, e_r.$$

  Remark. If $\operatorname{dom} f = \mathbb{P}^1 \times \mathbb{P}^1$, then the class $[f]$ can be defined similarly.

- **Example.** If $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by

$$x \mapsto (x_1^3 - x_1^2 x_0 : x_1^2 x_2 : x_1 x_2^2 : x_1 x_2 x_0 + x_2^3 - x_2^2 x_0),$$

  then $p_1 := (1 : 0 : 0)$, $p_2 := (1 : 1 : 0)$, $p_3 := (1 : 0 : 1)$ are its base points and

$$[f] = 3\, e_0 - 2\, e_1 - e_2 - e_3.$$

  Here $d = 3$, p_1 is a 2-fold base points and $f(p_1)$ "=" $(0 : 0 : 0 : 0) \notin \mathbb{P}^3$.

Base points and classes of rational functions 2/2

- **Recall.** If $\operatorname{dom} f = \mathbb{P}^2$, f has an m_i-fold base point at p_i for $1 \leq i \leq r$, then the **class** of f is defined as $[f] = d\, e_0 - m_1\, e_1 - \ldots - m_r\, e_r$, where d is the component degree of f.

- From a class $c := d\, e_0 - m_1\, e_1 - \ldots - m_r\, e_r$ we consider (if it exists) a basis for the vector space of all forms of degree d that have a m_i-fold base point p_i:

$$\langle f_0, \ldots, f_n \rangle$$

We shall denote $n+1$ by $h^0(c)$.
The **parametric map** of the class c is (up to $\operatorname{Aut} \mathbb{P}^n$) defined as

$$\Psi_c \colon \operatorname{dom} f \dashrightarrow \mathbb{P}^n, \quad x \mapsto \left(\tfrac{f_0}{q}(x) : \ldots : \tfrac{f_n}{q}(x) \right) \quad \text{where} \quad q := \gcd(f_0, \ldots, f_n).$$

- **Example.** Let $f \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ be as in the previous slide:

$$x \mapsto (x_1^3 - x_1^2 x_0 : x_1^2 x_2 : x_1 x_2^2 : x_1 x_2 x_0 + x_2^3 - x_2^2 x_0).$$

Recall that $[f] = 3\, e_0 - 2\, e_1 - e_2 - e_3$.
The parametric map $\Psi_{[f]} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ is (up to choice of basis) defined as

$$x \mapsto (x_1^3 - x_1^2 x_0 : x_1^2 x_2 : x_1 x_2^2 : x_1 x_2 x_0 : x_2^3 - x_2^2 x_0).$$

Notice that $h^0([f]) = 4 + 1 = 5$.

Compatible reducers 1/6

- **Recall.** Rational maps with domain in $\{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}$:

$$\mathcal{M} := \{f\colon \operatorname{dom} f \dashrightarrow \operatorname{img} f \subseteq \mathbb{P}^{\dim f} \mid \operatorname{dom} f \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}\}.$$

The **projective isomorphisms** for $f, g \in \mathcal{M}$:

$$\mathcal{P}(f,g) := \{\rho \in \operatorname{Aut} \mathbb{P}^{\dim f} \mid \rho(\operatorname{img} f) = \operatorname{img} g\}.$$

The **compatible reparametrizations** for $f, g \in \mathcal{M}$:

$$\mathcal{R}(f,g) := \{r \in \operatorname{bir}(\operatorname{dom} f, \operatorname{dom} g) \mid \rho \circ f = g \circ r \text{ for some } \rho \in \mathcal{P}(f,g)\},$$

where $\operatorname{bir}(\operatorname{dom} f, \operatorname{dom} g)$ is the set of birational maps between the domains.

- **Problem.** For given birational maps $f, g \in \mathcal{M}$ determine $\mathcal{P}(f,g)$.

- **Strategy.**
  Step 1: Determine a super set $\mathcal{S}$ of $\mathcal{R}(f,g)$.
  Step 2: Recover $\mathcal{P}(f,g)$ from $\mathcal{S}$.

- For step 1 we shall introduce the notion of "compatible reducers".

Compatible reducers 2/6

- A **condition** is a map
$$\mathbf{c}\colon \mathcal{M} \to \{0,1\},$$
where 0 and 1 are identified with False and True, respectively. Let
$$\mathcal{M}_{\mathbf{c}} := \{f \in \mathcal{M} \mid \mathbf{c}(f) = 1\}.$$

- A **reducer** consists of a condition $\mathbf{c}$ and a map
$$\mathbf{r}\colon \mathcal{M}_{\mathbf{c}} \to \mathcal{M}.$$

- A reducer $\mathbf{r}$ is **compatible** if $\mathbf{c}(f)$ is a projective invariant of $\operatorname{img} f \subset \mathbb{P}^{\dim f}$ for all $f \in \mathcal{M}$ and if for all $f, g \in \mathcal{M}_{\mathbf{c}}$ we have
$$\mathcal{R}(f,g) \subseteq \mathcal{R}(\mathbf{r}(f), \mathbf{r}(g)).$$

- **Goal.** Find a compatible reducer $\mathbf{r}$ such that computing a super set of $\mathcal{R}(\mathbf{r}(f), \mathbf{r}(g))$ is easier than computing a super set of $\mathcal{R}(f,g)$.

Compatible reducers 3/6

- **Recall.** We can associate to a map a class and vice versa as follows:

- The **class** of a map $f \in \mathcal{M}$ such that $\operatorname{dom} f = \mathbb{P}^2$ is defined as

$$[f] = d\, e_0 - m_1\, e_1 - \ldots - m_r\, e_r.$$

if f has an m_i-fold base point at p_i for $1 \leq i \leq r$ and $d = \deg f_0 = \ldots = \deg f_n$. The case when $\operatorname{dom} f = \mathbb{P}^1 \times \mathbb{P}^1$ is similar.

- The **parametric map** of a class

$$c := d\, e_0 - m_1\, e_1 - \ldots - m_r\, e_r$$

is (up to $\operatorname{Aut} \mathbb{P}^n$ and if it exists) defined as

$$\Psi_c \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^n, \quad x \mapsto \left(\tfrac{f_0}{q}(x) : \ldots : \tfrac{f_n}{q}(x) \right)$$

where $\langle f_0, \ldots, f_n \rangle$ is a basis of all forms on $\mathbb{P}^2$ of degree d that have a m_i-fold base point p_i and $q := \gcd(f_0, \ldots, f_n)$. We denote $n+1$ by $h^0(c)$.

Compatible reducers 4/6

- Let $[f] = d\,e_0 - m_1\,e_1 - \ldots - m_r\,e_r$ for $f \in \mathcal{M}$ with $m_i > 0$ for all $0 \leq i \leq r$.
  The **greatest common divisor** of f is defined as $\mathrm{gcd}[f] := \mathrm{gcd}(d, m_1, \ldots, m_r)$.
  The **canonical class** of f is defined as $\kappa_f := -3\,e_0 + e_1 + \ldots + e_r$.
  The product $-e_0^2 = e_j^2 = -1$ and $e_i \cdot e_j = 0$ for $0 \leq i < j \leq r$ defines an **intersection product between classes**.

- $\mathbf{r}_0(f) := \Psi_{[f]}$ and condition $\mathbf{c}_0(f)$ is defined as $\dim f < h^0([f]) - 1$.
  $\mathbf{r}_1(f) := \Psi_c$ with $c := [f] + \kappa_f$ and condition $\mathbf{c}_1(f)$ is defined as
  $$h^0(c) > 1 \quad \wedge \quad \neg\big([f]^2 > [\Psi_c]^2 = c \cdot [\Psi_c] = 0\big).$$
  $\mathbf{r}_2(f) := \Psi_b$ with $b := \frac{1}{\mathrm{gcd}[f]}[f]$ and $\mathbf{c}_2(f)$ is defined as $\mathrm{gcd}[f] > 1$.

- **Theorem.** The reducer $\mathbf{r}_i$ for $i \in \{0, 1, 2\}$ is compatible:
  If $(\mathbf{c}_i(f), \mathbf{c}_i(g)) = (1, 1)$, then
  $$\mathcal{R}(f, g) \subseteq \mathcal{R}(\mathbf{r}_i(f), \mathbf{r}_i(g)).$$
  If $\mathbf{c}_i(f) \neq \mathbf{c}_i(g)$, then $\mathcal{P}(f, g) = \emptyset$.
  If $(\mathbf{c}_i(f), \mathbf{c}_i(g)) = (0, 0)$, then $\mathrm{img}\, f$ is covered by either lines or conics (there are 5 **base cases**).

Compatible reducers 5/6

- **Theorem.** If $f \in \mathcal{M}$ such that

$$\mathbf{c}_0(f) = \mathbf{c}_1(f) = \mathbf{c}_2(f) = 0,$$

then either one of the following five **base cases** holds:

B1. $h^0([f]) = 3$, $[f]^2 = 1$ and $\operatorname{img} f \cong \mathbb{P}^2$.

B2. $h^0([f]) = 4$, $[f]^2 = 2$ and $\operatorname{img} f$ is a quadric surface.

B3. $h^0([f]) = [f]^2 + 1$, $1 \leq [f]^2 \leq 8$ and f defines a del Pezzo surface.

B4. $h^0(2\,[f] + \kappa_f) \geq 2$ and $\operatorname{img} f$ is a surface covered by lines.

B5. $h^0([f] + \kappa_f) \geq 2$ and $\operatorname{img} f$ is a surface covered by conics or lines.

- **Proposition.** If $f, g \in \mathcal{M}$ are both characterized by **B1**, then

$$\mathcal{R}(f,g) \subseteq \mathcal{S} := \{g^{-1} \circ r_c \circ f \mid c \in \mathcal{I}\} \quad \text{where} \quad \{r_c\}_{c \in \mathcal{I}} \cong \operatorname{Aut} \mathbb{P}^2.$$

- **Remark.** For the remaining cases **B2–B5** we can apply methods of J.G. Alcázar, M. Bizzarri, M. Hauer, C. Hermoso, M. Lávička, J. Vršek, . . . etc.

Compatible reducers 6/6

Algorithm.

- **input.** birational maps $f, g \in \mathcal{M}$ with domain in $\{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\}$.
- **output.** projective isomorphisms $\mathcal{P}(f,g) = \{\rho \in \operatorname{Aut} \mathbb{P}^{\dim f} \mid \rho(\operatorname{img} f) = \operatorname{img} g\}$.
- **method.**

Compute the base points of f and g.

$(\hat{f}, \hat{g}) \leftarrow (f, g)$

for $i \in \{0, 1, 2\}$ **do**

while $(\mathbf{c}_i(\hat{f}), \mathbf{c}_i(\hat{f})) = (1,1)$ **do** $(\hat{f}, \hat{g}) \leftarrow (\mathbf{r}_i(\hat{f}), \mathbf{r}_i(\hat{g}))$

if $(\mathbf{c}_i(\hat{f}), \mathbf{c}_i(\hat{f})) \neq (0,0)$ **return** $\emptyset$

Compute a super set $\mathcal{S}$ of $\mathcal{R}(\hat{f}, \hat{g})$ using the fact that $\hat{f}$ and $\hat{g}$ are both characterized by base case **B1**, **B2**, **B3**, **B4** or **B5**.

Recover $\mathcal{P}(f,g)$ from $\mathcal{S}$ by solving a system of alg. equations.

return $\mathcal{P}(f,g)$

The End.